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Travelling wave solution of variable-coefficient Burgers equation with variable- parameter tanh method

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Abstract—In this study, travelling wave solution of variable –coefficient Burgers equation is extracted using variable-parameter tanh method. Implementing this method, the used parameters are assumed to be functions of time. The solutions are found to be of hyperbolic type. For some special values of the parameters, the solitary wave solutions are found. Two particular solutions for some specific values of the parameters involved are plotted as illustrative examples. The approach used in this study is found to be effective and can be implemented in solving various nonlinear evolution equations with variable coefficients.

Index Terms— Burgers equation, Nonlinear evolution equation, Travelling wave, Variable- parameter tanh method, Solitary wave, Over-determined differential equations, Analytic solution.

1 Introduction

NON-linear evolution equations (NLEEs) are nonlinear partial differential equations (PDEs) having both spatial and time derivatives arising in modeling nonlinear physical phenomena that evolve over time. Basically, they are special types of PDEs and most of them have no analytical solutions. From the last few decades, NLEEs are studied more in seeking their exact solutions due to having availability of some symbolic computational software such as Maple, Mathematica, Matlab [1]. Recently, several analytical tools such as inverse scattering method [2], Hirota bilinear method [3], Painleve expansion method [4], sine-cosine method [5], homogeneous balance method [6], homotopy perturbation method [7-9], Adomian decomposition method [10], tanh function method [11-16], F-expansion method [17-21], exp-function method [22, 23], auxiliary equation method [24], (G'/G)-expansion method [25-31], simplest equation method [32], etc. have been developed in searching analytical solutions of NLEEs. Most of the aforementioned methods are developed due to solve NLEEs with constant coefficients using constant parameters. Recently, in some studies such as [16, 29, 33-36], it is found to be used variable parameter method in solving variable coefficient NLEEs. In [29, 33], variable coefficient KdV equation is solved using variable parameter (G'/G)-expansion method. These studies make our interest to see the solution of NLEEs with variable coefficients using variable parameter tanh method. It is to be mentioned here that tanh method was introduced in Huibin and Kelin [37] and further development was made through several studies [38-43]. In all of the mentioned studies, the constant coefficient NLEEs are solved using constant parameter tanh method. It is to be mentioned at this junction that Tian et al. [44] in his study claimed that 'the generalized tanh method can be extended from the situation with coefficient constants to that with coefficient functions'. Keeping this idea in mind, Zhang and Zhang [16] used variable parameter tanh method in solving Burgers equation with variable coefficients. In the present study, we intend to solve Burgers equation with variable coefficients with variable parameter tanh method.

The rest of the paper is organized as follows. In section 2, the method is discussed briefly, section 3 deals with the implementation of the variable-parameter tanh method in solving variable–coefficient Burgers equation with variable coefficients. Discussion of results and conclusion are presented in section 4.

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2. A BRIEF DESCRIPTION OF THE METHOD

A NLEE with state variable $u(X)$, where $X=(x,y,z,t)$ can be defined in following form:

$$F(u,u_t,u_x,u_y,u_z,u_{xt},u_{yt},u_{zt},u_{tt},u_{xx},u_{yy},u_{zz},\ldots)=0, \quad (1)$$

where the suffixes indicate derivatives.

The solution of Eq. (1) can be expressed as a polynomial of $\tanh(X)$ as follows:

$$u(X)=\sum_{i=0}^{n}\alpha_i(X)Y^i,\ \alpha_n(X)\neq 0, \quad (2)$$

where $Y=\tanh(\xi)$ while $\xi=\xi(X)$, $X=X(x,y,z,t)$

represents independent variable and the coefficients of the polynomial $\alpha_i(t),\ i=1, 2, ..., n$ are assumed to be functions of time instated of constants. The consideration $Y=\tanh(\xi)$ leads to take the differential operators in following forms:

$$\frac{d}{d\xi}=(1-Y^2)\frac{d}{dY};$$

$$\frac{d^2}{d\xi^2}=-2Y(1-Y^2)\frac{d}{dY}+(1-Y^2)^2\frac{d^2}{dY^2}$$ and so on.

To determine $u(X)$ explicitly, it is required to follow the following steps:

Step 1. Determine the value of n by homogeneous balancing between highest order derivative and nonlinear terms present in the NLEE characterized by Eq. (1).

Step 2. Substitute Eq. (2) in Eq. (1) and collect the coefficients as an order of Y. Thus a polynomial will be obtained with Y. Equating the coefficients of the polynomial to zero, a system of over-determined differential equations with $\alpha_i(t),\ i=1, 2, ..., n$ and ξ is obtained.

Step 3. Solve the system of over-determined differential equations obtained in *Step 2* for getting the values of parameters $\alpha_i(t)$ and ξ by using the symbolic software Maple, Mathematica or Matlab.

Step 4. Use the computed values of the parameters in Eq. (2). Then the exact travelling wave solution of the given NLEEs characterized by Eq. (1) will be obtained.

3. IMPLEMENTATION OF THE VARIABLE PARAMETER TANH METHOD IN SOLVING BURGERS EQUATION

Burgers equation is one of the simplest NLEEs which is named after Burgers that arose in studying turbulence in 1939 [45]. The one dimensional variable coefficients Burgers equation can be written as [33]

$$u_t-d(t)u_{xx}+a(t)uu_x=0, \quad (3)$$

where $d(t)$ and $a(t)$ are functions of time.

After balancing between highest order derivative and nonlinear terms present in Eq. (3), we have $n=1$. Thus Eq. (3) has the solution having the following form:

$$u=\alpha_0(t)+\alpha_1(t)Y,\ \ \alpha_1(t)\neq 0, \quad (4)$$

where $Y=\tanh(\xi)$ stratifying Eq. (3) with $\xi=p(t)x+q(t)$ while the functions $p(t)$ and $q(t)$ are to be determined.

Equation (3) turns into a polynomial of Y with the help of Eq. (4). Equating the coefficients of order of Y to zero, the following set of over-determined differential equations is obtained:

$$x^0Y^3:\ -2d(t)\alpha_1(t)p(t)^2-p(t)\alpha_1(t)p(t)^2a(t)=0,$$

$$x^0Y^2:-p(t)a(t)\alpha_1(t)\alpha_0(t)-\alpha_1(t)\left(\frac{d}{dt}q(t)\right)=0,$$

$$x^0Y^1:2d(t)\alpha_1(t)p(t)^2+p(t)\alpha_1(t)^2a(t)+\left(\frac{d}{dt}\alpha_1(t)\right)=0,$$

$$x^0Y^0:\ p(t)\alpha_1(t)a(t)\alpha_0(t)+\alpha_1(t)\left(\frac{d}{dt}q(t)\right)+\left(\frac{d}{dt}\alpha_o(t)\right)=0, \quad (5)$$

$$x^1Y^0:\alpha_1(t)\left(\frac{d}{dt}p(t)\right)=0,$$

$$x^1Y^2:-\alpha_1(t)\left(\frac{d}{dt}p(t)\right)=0.$$

Solving the system of over-determined differential equations represented by Eq. (5), we have the following values of the parameters: $p(t)=P$, $\alpha_0(t)=Q$, $\alpha_1(t)=R$, $a(t)=-\frac{2d(t)P}{R}$ and $q(t)=\int_0^t\frac{2P^2Q\,d(t)}{R}dt$, where P, Q and R are arbitrary constants.

Thus $\xi=Px+q(t)$, where $q(t)=\int_0^t\frac{2P^2Q\,d(t)}{R}dt$.

Now substituting the computed values of the parameters in Eq. (4), it turns into the following form:

$$u(x,t)=Q+R\tanh\left(Px+\frac{2P^2Q}{R}\int_0^t d(t)dt\right). \quad (6)$$

This is the obtained solution of the variable coefficient Burgers equation represented by Eq. (3).

4. RESULT DISCUSSION AND CONCLUSION

In this study, the variable parameter $\tanh$ method is used in solving one dimensional variable coefficient Burgers equation. The obtained result is found to be similar with one of the results obtained in [33]. If it is assumed that the coefficients are constants then the computed result seems to be quite similar with the ones obtained in [46, 47]. The obtained solution symbolized by Eq. (6) is a solitary wave type solution. The computed solutions are also depicted in Figs. 1-4 for choosing different types of functions (constant, algebraic, exponential and trigonometric) for $d(t)$ with the fixed values of other parameters as illustrative examples. From the computed results, it can be pointed out here that for choosing any types of functions for $d(t)$, the computed results were found to be of solitary wave type. However, the method used here is a more general approach in solving some NLEEs with variable coefficients. So, it can be an alternative approach in solving NLEEs with variable coefficients.

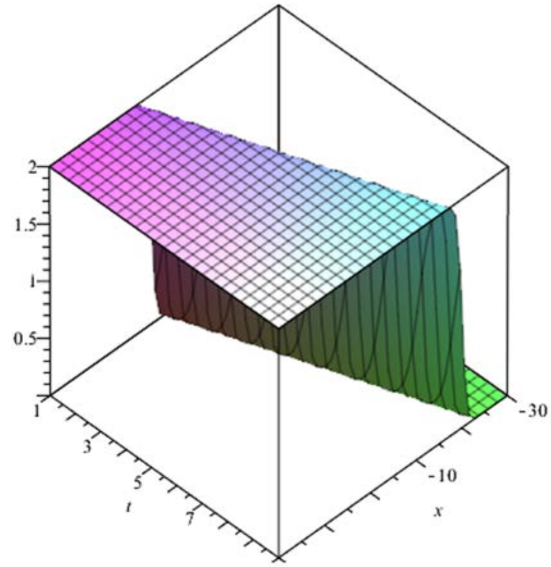

Fig. 1 Computed solutions for $P=1,\ Q=1,\ R=1$ where $d(t)$ is a constant function $[d(t)=1]$.

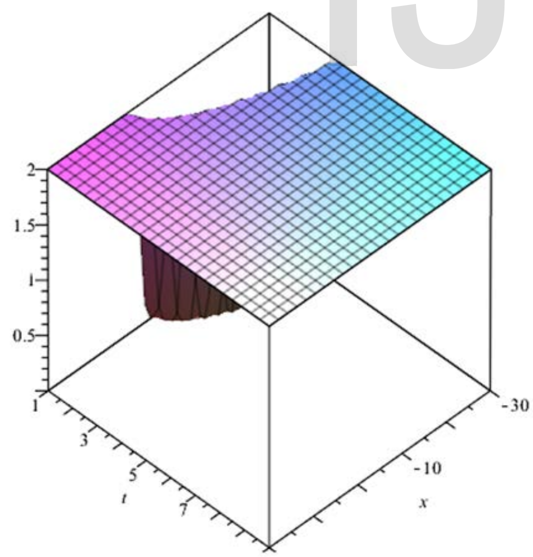

Fig. 2 Computed solutions for $P=1,\ Q=1,\ R=1$ where $d(t)$ is an algebraic function $[d(t)=t^2]$.

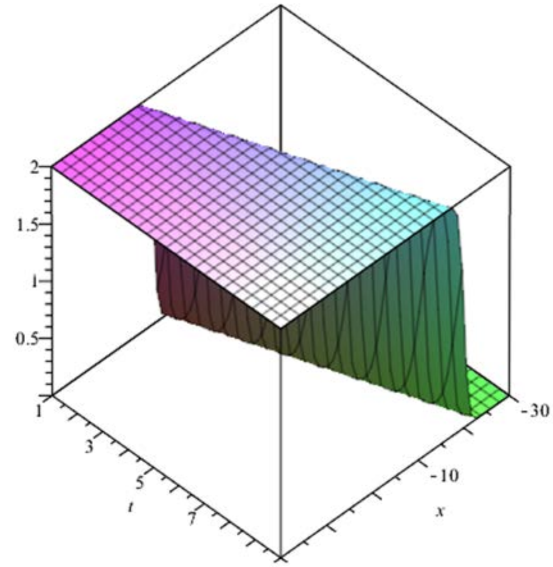

Fig. 3 Computed solutions for $P=1,\ Q=1,\ R=1$ where $d(t)$ is an exponential function $[d(t)=e^{-2t}]$.

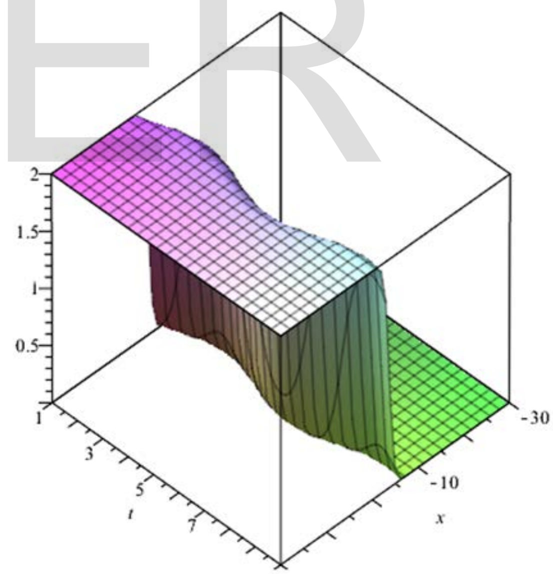

Fig. 4 Computed solutions for $P=1,\ Q=1,\ R=1$ where $d(t)$ is a trigonometric function $[d(t)=\sin t]$.

ETHICAL STATEMENT

The author declare that they have no conflict of interest.

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